

Electron Scattering by a Solenoid

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Abstract

The quantum-mechanical problems of electron scattering by an infinitely thin solenoid and by a half of an infinitely thin solenoid are examined from the viewpoint of constructing a self-adjoint Hamiltonian. It is demonstrated that in both problems there exist unique self-adjoint operators with a “non-singular” domain, that, due to physical reasons, are identified with the corresponding Hamiltonians. In the case of quantized values of magnetic flow along the solenoid, the electron does not experience any scattering by the string. It is shown that the scattering amplitude and wave function of an electron in the problem of scattering by an infinitely long solenoid of radius a in the limit $a \rightarrow 0$ turn into the corresponding expressions for the problem of an infinitely thin solenoid. In particular, at a quantized value of magnetic flow along the solenoid, scattering disappears at $a \rightarrow 0$.

1 Introduction

In this article, we examine two quantum-mechanical problems: the scattering of an electron by the field of an infinite, infinitely thin solenoid, $A_k^{(c)}(\vec{r}) = -\mu \frac{\varepsilon_{3ki} r_i}{\rho^2}$, $\vec{r} = (x, y, z)$, $\rho^2 = x^2 + y^2$, and the scattering of an electron by the field of a semi-infinite, infinitely thin solenoid, $A_k^{(D)}(\vec{r}) = -\mu \frac{\varepsilon_{3ki} r_i}{r(r-z)}$. Our interest in these problems is due to the following reasons.

As is well-known, Dirac [1] proposed a Lagrangian theory of electrodynamics with two charges: the electric and magnetic ones, subject to a charge quantization condition; see [1]. In Dirac’s theory, however, the electromagnetic potentials contain singularities on certain lines (Dirac’s strings) coming out of the magnetic charges (in principle, Dirac’s strings may also come out of the electric charges, as well as out of the charges of both types). The motion of these strings is not determined by the Lagrangian and may be defined arbitrarily. Dirac presented some arguments in favour of the fact that physical results must actually be independent of the form and specific motion of strings, since the electromagnetic potentials for two different positions of a string are related by a gradient transformation (anywhere outside strings). This argument is not very convincing, because, in fact, two potentials for two different positions of strings cannot be related by a gradient transformation in the entire space (for more details, see [1]).

The simplest problem concerning the interaction of an electron with a magnetic charge is the problem of the scattering of a non-relativistic electron by the static field of an infinitely heavy magnetic charge. In addition, the electromagnetic potential (suggested by Dirac) of a magnetic charge is identical with the electromagnetic potential of a semi-infinite, infinitely thin solenoid. According to Dirac’s hypothesis, physical results (in particular, the cross-section of electron scattering) must not depend on the direction of a string (the above expression for $A_k^{(D)}$ corresponds to the rectilinear form of a string; of course, physical results must not depend on the form of a string, either). Such a problem has been examined by numerous authors [1]. Finally, in the work by Zwanziger [2] it was found that the cross-section of electron scattering by the potential of a magnetic charge is, in fact, independent of the direction of a string.

If the electron “does not feel” the magnetic field along the string, then it must not experience any scattering by the potential of an infinitely thin and infinitely long solenoid (in case the magnetic flow along the string takes quantized values, which is in agreement with the quantization condition for the electric and magnetic charges). This problem has been examined (in a different connection) by Aharonov and Bohm [3], who have found that in the case of quantized values of the magnetic charge of a solenoid (this corresponds to $\mu = 0, \pm 1, \pm 2, \dots$ in the above expression for $A_k^{(c)}$), the electron, indeed, does not experience any scattering by the string.

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If, however, one looks at the structure of solutions of this problem, it turns out that the wave function of an electron has the form $\psi = e^{i\mu\varphi} e^{i\vec{k}\vec{r}}$, where \vec{k} is the momentum of a moving electron, and φ is the angle between the projections of the vectors \vec{k} and \vec{r} on the plane xy . This form of a wave function suggests the idea that the potential $A_k^{(c)}$ is a pure gradient: $A_k^{(c)} \sim \mu \partial_k \varphi$. If, indeed, one formally (i.e., without taking into account the fact that φ is a discontinuous function) calculates the gradient of φ , then one obtains the equality $A_k^{(c)} = \mu \partial_k \varphi$. It is clear, nonetheless, that such an equality cannot take place, since the rot of its r.h.s. is equal to zero, while at the same time $\text{rot } A_k^{(c)}$ is the magnetic field (with a finite flow μ) along the string. Then, there arises the question as to the correctness of the solution presented by Aharonov and Bohm for this problem.

In the case of electron scattering by a semi-infinite, infinitely thin solenoid, Zwanziger [2] has found that the scattering amplitude has the form $f(\vec{k}, \vec{k}') = e^{i\Omega} f_0(\vec{k}, \vec{k}')$ (\vec{k} and \vec{k}' are the respective initial and final momenta of an electron; f_0 is the scattering amplitude in case the string is directed along the momentum of a moving electron) and depends, as a consequence, merely on the angle between \vec{k} and \vec{k}' ; Ω is a certain function (see Section 4). As will be shown in Section 4, the wave function $\psi = e^{i\Omega} \psi_0$ has the same form, where ψ_0 is the wave function in case the string is directed along \vec{k} . Once again, there arises the suspicion that such a form of solution is due to the fact that the difference of vector potentials $A_i^{(D)} - A_i^{(D,k)} (A_i^{(D,k)}(\vec{r}) = -\mu \frac{\varepsilon_{ilj} k_l r_j}{r(kr - \vec{k}\vec{r})})$ is the potential of a semi-infinite, infinitely thin solenoid, with the string being directed along \vec{k} is a pure gradient. If one formally computes the gradient of Ω (i.e., without taking into account the fact that Ω is a discontinuous function), then one obtains $A_i^{(D)} - A_i^{(D,k)} = \partial_i \Omega$. However, this equality cannot take place, either, because the rot of the r.h.s. is equal to zero, while at the same time the rot of the l.h.s. is the flow of magnetic field (with finite flows) along the strings. Thus, in this case there also arises the question as to the correctness of the solution.

The question of correctness of the solutions found by Aharonov–Bohm and Zwanziger for the problems in question arises also in connection with another reason. The point is that the Hamiltonians of these problems are extremely singular and thus cannot be defined immediately in the class of functions for which they would be self-adjoint operators (in particular, the natural domain of a Hamiltonian does not include all differentiable functions). In the natural domain, the Hamiltonians prove to be merely symmetric, with non-vanishing deficiency indices. In addition, it is well-known [4, 5] that the problem of constructing a self-adjoint Hamiltonian from a given symmetric operator (a self-adjoint extension of a symmetric operator) admits more than one solution. Let us remind that the self-adjoint character of an operator is necessary for the corresponding operator of evolution to be unitary and uniquely defined in the entire Hilbert space. Different self-adjoint extensions lead to different solutions of the scattering problem, corresponding to the same Schrödinger equation.

In Sections 2 and 4, in the cases of infinite and semi-infinite solenoids respectively, with arbitrary values of μ , it is shown that in both problems there exist unique self-adjoint extensions of the Hamiltonian, such that they obey the usual physical condition: the Hilbert-space functions for which the Hamiltonian is defined must not be singular (by the way, it is natural to call this condition the “principle of minimal singularity”). Given this, it is shown that for the integer values of the parameter μ any scattering by the string is absent and the solution of the scattering problem is identical with the corresponding solutions of Aharonov–Bohm and Zwanziger. Therefore, in both problems it is rigorously proved that in the case of a quantized magnetic flow along the strings the electron does not experience scattering, in agreement with Dirac’s hypothesis, and that, in consequence, a semi-infinite, infinitely thin solenoid can actually be considered as a realization of a fixed monopole (magnetic charge). In the same sections, it is shown how to solve the (seeming) difficulty that has been described earlier in this Introduction, i.e., the one related with the gauge transformations of potentials and with the form of solutions.

In Section 3, it is shown that the result of Aharonov–Bohm can be deduced with the help of a “physical” regularization of the potential, which consists in using a solenoid with a finite radius a , rather than an infinitely thin solenoid, and then making a go to zero.

In Appendix A, in connection with the article [7], it is shown that the problem of scattering by the potential of an infinitely thin solenoid cannot be solved as a perturbation theory in $\Delta\mu$, where $\mu = [\mu] + \Delta\mu$, and $[\mu]$ is the integer part of μ .

2 Electron scattering by the field of an infinitely thin and infinitely long solenoid

The electromagnetic potential $A_k^{(c)}$ of an infinitely thin and infinitely long solenoid has the form

$$A_k^{(c)}(\vec{r}) = -\mu \frac{\varepsilon_{3ki} r_i}{\rho^2}. \quad (1)$$

It is easy to see that outside the z axis the magnetic field $\vec{H} = \text{rot } \vec{A}^{(c)}$ is equal to zero everywhere. Nevertheless, the flow $\oint \vec{A}^{(c)} d\vec{l}$ through any surface with the boundary being any contour (even an infinitely small one) around the z axis does not vanish and is equal to μ .

The Schrödinger equation for the scattering of an electron by this same potential has the form

$$i\partial_t \psi(\vec{r}, t) = H\psi(\vec{r}, t), \quad (2)$$

where

$$H = \left(-i\partial_k + A_k^{(c)}(\vec{r}) \right)^2, \quad (3)$$

and the mass of an electron is set equal to $1/2$.

Eq. (2) has the following solution:

$$\psi(\vec{r}, t) = \exp\{-ik_z^2 t + ik_z z\} \psi(\vec{\rho}, t), \quad (4)$$

where k_z is the projection of the momentum of the electron on the z axis, $\vec{\rho} = (x, y)$, and $\psi(\vec{\rho}, t)$ obeys the equation

$$i\partial_t \psi(\vec{\rho}, t) = \tilde{H}\psi(\vec{\rho}, t), \quad (5)$$

$$\begin{aligned} \tilde{H} &= -\Delta + \frac{2\mu i}{\rho^2} (y\partial_x - x\partial_y) + \frac{\mu^2}{\rho^2} \\ &= -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2} (-\partial_\varphi^2 - 2i\mu\partial_\varphi + \mu^2). \end{aligned} \quad (6)$$

Let us now construct a self-adjoint Hamiltonian corresponding to the differential expression (6).

It is clear that \tilde{H} cannot be determined immediately for any differentiable (and decreasing sufficiently fast as $\rho \rightarrow \infty$) functions that belong to L_2 (L_2 is the Hilbert space of square-integrable functions $\psi(\vec{\rho})$) because of the terms $\frac{\mu^2}{\rho^2}$ and $\frac{1}{\rho^2} (y\partial_x - x\partial_y)$.

Let us define an operator H_0 as a differential operator (6) on the domain D_0 , where D_0 is the set of twice-differentiable functions with a compact support that turn to zero at $\rho < a$ for a certain a (a may be different for different functions). D_0 is dense in L_2 ; (6) is obviously defined for any $\psi \in D_0$, and for any $\psi_1, \psi_2 \in D_0$ there holds the equality

$$\int d\vec{\rho} \psi_1^* (H_0 \psi_2) = \int d\vec{\rho} (H_0 \psi_1^*) \psi_2. \quad (7)$$

Therefore, H_0 is a symmetric operator [4, 5]. Besides, H_0 is a positive operator.

In order to construct self-adjoint extensions of the operator H_0 , one should obtain the eigenfunctions $U^{(\pm)}$ of the adjoint operator H_0^* with the eigenvalues $\pm i$. In addition, H_0^* is defined as the differential expression (6); however, the domain is no longer restricted by the boundary values at $\rho = 0$ or at $\rho = \infty$. This implies that the function $U^{(+)}$ (corresponding to the eigenvalue $+i$) must obey the equation

$$\tilde{H}U^{(+)} = iU^{(+)} \quad (8)$$

on the entire plane, except, perhaps, the point $\rho = 0$. Let us search for the eigenfunctions U in the form

$$U^{(+)}(\vec{\rho}) = e^{im\varphi} U_m^{(+)}(\vec{\rho}). \quad (9)$$

Note that the function $e^{im\varphi} = \left(\frac{x+iy}{\rho}\right)^m$ is infinitely differentiable everywhere except the origin. Therefore, the action of the operator \tilde{H} on a function of the form (9) (which ought to be known everywhere except the origin) does not require any redefinition. The function $U_m^{(+)}$ satisfies the equation

$$\left(-\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{(m+\mu)^2}{\rho^2}\right) U_m^{(+)}(\rho) = iU_m^{(+)}(\rho). \quad (10)$$

Eq. (10) is the Bessel equation. Since (10) must take place at $\rho \neq 0$, it does not require its redefinition neither in Cartesian nor in cylindric coordinates. A solution of (10) being square-integrable at ∞ is given by

$$U_m^{(+)}(\rho) = N_m^{-1} H_{m+\mu}^{(1)}(e^{i\frac{\pi}{4}}\rho), \quad (11)$$

where N_m is a normalization factor (in case the function is square-integrable). Function (11) is square-integrable at zero (with respect to the measure $dx dy \sim \rho d\rho$) only if $|m+\mu| < 1$. Therefore, the operator H_0^* has two eigenfunctions that belong to L_2 with the eigenvalue i at a non-integer μ and one eigenfunction [...] an integer μ . H_0^* has the same number of eigenfunctions with the eigenvalue $-i$. This follows from the solution of an equation similar to (10),

$$U_m^{(-)}(\rho) = N_m^{-1} H_{m+\mu}^{(2)}(e^{-i\frac{\pi}{4}}\rho), \quad (12)$$

as well as from the fact that H_0 is a positive operator.

Therefore, H_0 is a symmetric operator with non-vanishing (and equal) deficiency indices: $(2, 2)$ at $\Delta\mu \neq 0$ and $(1, 1)$ at $\Delta\mu = 0$, where $\mu = -[\mu] - \Delta\mu$, $-[\mu]$ being the integer part of μ .

All essentially self-adjoint extensions H_θ of the operator H_0 are described as follows [4, 5]:

a) the domain is

$$D_\theta = D_0 + F + F^\theta, \quad (13)$$

where

$$F = \begin{cases} A_1 e^{i([\mu]+1)\varphi} N_{[\mu]+1}^{-1} H_{1-\Delta\mu}^{(1)}(e^{i\frac{\pi}{4}}\rho) + A_2 e^{i[\mu]\varphi} N_{[\mu]}^{-1} H_{\Delta\mu}^{(1)}(e^{i\frac{\pi}{4}}\rho), & \text{at } \Delta\mu \neq 0 \\ A e^{i[\mu]\varphi} H_0^{(1)}(e^{i\frac{\pi}{4}}\rho), & \text{at } \Delta\mu = 0, \end{cases} \quad (14)$$

$$F^\theta = \begin{cases} B_1 e^{i([\mu]+1)\varphi} N_{[\mu]+1}^{-1} H_{1-\Delta\mu}^{(2)}(e^{-i\frac{\pi}{4}}\rho) + B_2 e^{i[\mu]\varphi} N_{[\mu]}^{-1} H_{\Delta\mu}^{(2)}(e^{-i\frac{\pi}{4}}\rho), & \text{at } \Delta\mu \neq 0 \\ A e^{i\theta} e^{i[\mu]\varphi} \rho H_0^{(2)}(e^{-i\frac{\pi}{4}}\rho), & \text{at } \Delta\mu = 0, \end{cases} \quad (15)$$

and $B_i = A_j \theta_{ji}$, A_i , A are arbitrary complex numbers; θ_{ij} and θ are, respectively, an arbitrary (although fixed for a given extension) unitary matrix and real number;

b)

$$H_\theta D_\theta = H_0 D_0 + iF - iF^\theta. \quad (16)$$

Self-adjoint extensions are given by the closure: $H_\theta : \overline{H}_\theta = H_\theta^*$. In addition, the domain of \overline{H}_θ is $\overline{D}_\theta = \overline{D}_0 + F + F^\theta$, where \overline{D}_θ is the domain of the operator $\overline{H}_0 = H_0^{**}$. The functions $\psi \in \overline{D}_0$ are non-singular.²

In order to select a self-adjoint extension, we require that the functions from the domain of a “physical” operator should be nonsingular. This fixes the extension in a unique way:

$$\theta_{ij} = \begin{vmatrix} \exp\left\{-i(1-\Delta\mu)\frac{\pi}{4}\right\} & 0 \\ 0 & \exp\left\{-i\Delta\mu\frac{\pi}{4}\right\} \end{vmatrix}, \text{ for } \Delta\mu \neq 0, \quad (17)$$

$$\theta = 0, \text{ for } \Delta\mu = 0. \quad (18)$$

We must now obtain the complete system of generalized eigenfunctions for the self-adjoint operator (which shall be denoted as \mathcal{H}), corresponding to (17) and (18). Since the matrix θ_{ij} is diagonal, the subspaces of functions of the form $\left(\frac{x+iy}{\rho}\right)^m f(\rho)$ reduce \mathcal{H} , and we must find generalized eigenfunctions of a “nonsingular” extension of Bessel’s differential expression on a semiaxis. This problem has been solved in [6], where all the inversion formulas related to Bessel’s differential expression of an arbitrary

²The functions $\psi \in \overline{D}_0$ become continuous after their correction on the zero-measure set $\psi(0) = 0$; see [5].

index $\nu \geq 0$ have been described. In the case under consideration, the generalized eigenfunctions are given by

$$e^{im\varphi} J_{|m+\mu|}(\sqrt{\lambda}\rho), \quad m = 0, \pm 1, \pm 2, \dots, \quad (19)$$

where λ is an arbitrary real number, and the operator \mathcal{H} proves to be positive. We note that in case $\Delta\mu = 0$ the operator \mathcal{H} also proves to be the only positive extension of the operator H_0 .

Thus, it has been shown that under the physical condition of “minimal singularity”, i.e., the condition that the domain of the Hamiltonian should be composed entirely of nonsingular functions, there exists a unique self-adjoint operator \mathcal{H} related to the differential expression (3). It is natural to refer to the operator \mathcal{H} as the Hamiltonian of the problem in question. The complete system of generalized eigenvectors of this operator is given by the set of functions (19).

Let us now solve the scattering problem, i.e., construct a wave function $\psi_k(\vec{\rho})$ subject to the Schrödinger equation

$$\mathcal{H}\psi_k(\vec{\rho}) = k^2\psi_k(\vec{\rho}), \quad k^2 = E - k_z^2 \quad (20)$$

(with E being the total energy of the electron; \vec{k} being the projection of the total momentum of the electron on the xy plane), as well as to the following asymptotic condition:

$$\psi_k(\vec{\rho}) \xrightarrow{\rho \rightarrow \infty} e^{i\vec{k}\vec{\rho}} + \frac{f(\varphi)}{\sqrt{2\pi k\rho}} e^{ik\rho - i\frac{\pi}{4}}. \quad (21)$$

Condition (21) implies that in case $\rho \rightarrow \infty$ the wave function is a superposition of an incoming plain wave and a scattered divergent cylindrical wave; $f(\varphi)$ is the scattering amplitude.

Using the expansion

$$e^{i\vec{k}\vec{\rho}} \rightarrow \frac{1}{\sqrt{2\pi k\rho}} \left[e^{ik\rho - i\frac{\pi}{4}} \sum_{m=-\infty}^{\infty} e^{im(\varphi-\delta)} + e^{-ik\rho + i\frac{\pi}{4}} \sum_{m=-\infty}^{\infty} (-)^m e^{im(\varphi-\delta)} \right], \quad (22)$$

where δ is the angle between the vector \vec{k} and the x axis, we obtain, due to the well-known asymptotic behavior of Bessel's functions,³

$$\psi_k(\vec{\rho}) = e^{-i\mu\frac{\pi}{2}} \sum_{m+\mu \geq 0} e^{im(\varphi-\delta)} i^m J_{m+\mu}(k\rho) + e^{-i\mu\frac{\pi}{2}} \sum_{m+\mu < 0} e^{im(\varphi-\delta)} (-i)^m J_{|m+\mu|}(k\rho). \quad (23)$$

The scattering amplitude ($\varphi - \delta \neq 0$) equals to

$$f(\varphi) = e^{i([\mu] + \frac{1}{2})(\varphi-\delta)} (-)^{[\mu]} \frac{\sin^2 \Delta\mu\pi}{\sin^2\left(\frac{\varphi-\delta}{2}\right)}. \quad (24)$$

The cross-section of scattering equals to ($\varphi - \delta \neq 0$)

$$\sigma \sim |f(\psi)|^2 = \frac{\sin^2 \Delta\mu\pi}{\sin^2\left(\frac{\varphi-\delta}{2}\right)}, \quad (25)$$

which is identical with the result of [3]. Therefore, it follows from (25) that in the case of a quantized value of flow along an infinite, infinitely thin solenoid (μ being integer) the scattering on the potential is absent.

Let us make some more remarks concerning the case of an integer μ . Namely, we suppose $\mu = n$. The wave function then admits the representation

$$\psi_k = (-)^n e^{-in(\varphi-\delta)} e^{-i\vec{k}\vec{\rho}}. \quad (26)$$

³Note that the wave function (23) and the scattering amplitude (24) are single-valued functions of the coordinates, whereas the wave function of Aharonov and Bohm is not single-valued. This is related to the fact that, instead of the asymptotic condition (21), the authors of [3] use a condition that differs from (21) by the multiplier $e^{i\mu\varphi}$, which results in the non-single-valued character of the wave function in [3]. The squared module of the scattering amplitude in [3], however, is identical with the squared module of function (24).

(26) has the following asymptotics:

$$\begin{aligned}\psi_k(\vec{\rho}) &\rightarrow \frac{1}{\sqrt{2\pi\rho}} \left[(-)^n \delta(\psi - \delta) e^{ik\rho - i\frac{\pi}{4}} + \delta(\varphi - \delta - \pi) e^{-ik\rho + i\frac{\pi}{4}} \right] \\ &= e^{i\vec{k}\vec{\rho}} + ((-)^n - 1) \delta(\varphi - \delta) \frac{e^{ik\rho - i\frac{\pi}{4}}}{\sqrt{2\pi k\rho}}.\end{aligned}\quad (27)$$

(27) shows that scattering, in pure form, is absent only for even μ . For odd μ , however, the electron “senses” the presence of the string.

Another remark is as follows. The wave function (26), as mentioned in Introduction, contains the multiplier $e^{-in\varphi}$, being a cause for a certain perplexity, which has been discussed in Introduction. A solution of the mentioned paradox is given by the fact that for integer μ the potential $A_k^{(c)}$ can actually be transformed out of the Schrödinger equation with the help of a *gauge transformation*, which, however, is not a *gradient transformation*. Namely, the potential $A_k^{(c)}$ can be represented in the form (for integer $\mu = n$)

$$A_k^{(c)} = -ie^{in\varphi} \partial_k e^{-in\varphi} \equiv -i \left(\frac{x + iy}{\rho} \right)^n \partial_k \left(\frac{x - iy}{\rho} \right)^n. \quad (28)$$

For non-integer μ , such a representation *does not take place*!

It can be readily verified that (28) holds true in the entire space, except, perhaps, the origin. Representation (28) turns out to be sufficient to eliminate the potential from the Schrödinger equation with the help of the gauge transformation

$$A_k \rightarrow A_k - iU^{-1} \partial_k U, \quad \psi \rightarrow U\psi. \quad (29)$$

In order that the transformation of A_k be a gradient one, it is necessary to impose the subsidiary condition of a continuous differentiability of $\ln U$, which is not necessarily fulfilled in the general case. This is precisely what happens in the problem under consideration. Representation (28) is not reduced to a pure gradient, since in this case $\ln U = -i\varphi$ is a discontinuous function (for instance, along the line $y = 0, x > 0$).

For the sake of completeness, one should investigate the point $\rho = 0$ more accurately. We will, however, restrict ourselves to the above qualitative remark.

3 A thin solenoid as a limit of a thick solenoid

Let us replace the potential $A_k^{(c)}$ in expression (3) by a regularized potential $A_k^{(R)}$:

$$A_k^{(R)}(\vec{r}) = A_k^{(c)}(\vec{r}) f_a(\rho), \quad (30)$$

$$f_a(\rho) = 1, \quad \rho \geq a; \quad f_a(0) = 0; \quad (31)$$

besides, f_a is continuous at the point $\rho = a$. Note that the flow of magnetic field corresponding to the potential (30) is equal to μ .

Let us select the eigenfunctions of H as solutions being regular at zero, which is usually postulated on the basis of physical reasons.⁴ These functions are

$$U_k^{(m)}(\vec{\rho}) = \begin{cases} c_m F_{m,k}(\rho) e^{im\varphi}, & \rho \leq a, \\ c_m (A_m J_{|m+\mu|}(k\rho) + B_m N_{|m+\mu|}(k\rho)) e^{im\varphi}, & \rho \geq a, \end{cases} \quad (32)$$

The function $F_{m,k}(\rho)$ is a solution, regular at zero (normalized as $\rho^{|m|}$ when $\rho \rightarrow 0$), of the following equation:

$$\left[-\partial_\rho^2 - \frac{1}{\rho} \partial_\rho + \frac{1}{\rho^2} (f_a(\rho) \mu^2 + 2m\mu f_a(\rho) + m^2) \right] F_{m,k}(\rho) = k^2 F_{m,k}(\rho). \quad (33)$$

⁴From the mathematical viewpoint, one can say that the Hamiltonian, defined as the differential expression (3) with potential (30) and a domain consisting of the set of all the twice-differentiable functions with a compact support, is an essentially self-adjoint operator. This operator is also a unique self-adjoint operator with a “nonsingular” domain corresponding to the differential expression (3) with potential (30).

The coefficients c_m are arbitrary, whereas the coefficients A_m and B_m are chosen from the condition that $U_k^{(m)}$ and their first derivatives be continuous at the point $\rho = a$:

$$A_m = \frac{\pi a}{2} \left[F_{m,k}(a) N'_{|m+\mu|}(ka) - F'_{m,k}(a) N_{|m+\mu|}(ka) \right], \quad (34)$$

$$B_m = \frac{\pi a}{2} \left[F'_{m,k}(a) J_{|m+\mu|}(ka) - F_{m,k}(a) J'_{|m+\mu|}(ka) \right]. \quad (35)$$

The solution $\Psi_k(\vec{\rho})$ of the Schrödinger equation with asymptotics (21) is given by

$$\Psi_k(\vec{\rho}) = \psi_k(\vec{\rho}) - i \sum_m e^{im(\varphi-\delta)+im\pi-i|m+\mu|\frac{\pi}{2}} \frac{b_m}{1+ib_m} H_{|m+\mu|}^{(1)}(k\rho), \quad \rho \geq a, \quad (36)$$

$$\Psi_k(\vec{\rho}) = \sum_m \frac{(-)^m e^{im(\varphi-\delta)-i|m+\mu|\frac{\pi}{2}} F_{m,k}(\rho)}{1+ib_m} \frac{F_{m,k}(\rho)}{A_m}, \quad \rho \leq a. \quad (37)$$

The corresponding scattering amplitude ϕ_k equals to ($\varphi - \delta \neq 0$)

$$\phi_k = f(\varphi) - 2i \sum_m \frac{b_m}{1+ib_m} (-)^m e^{im(\varphi-\delta)-i|m+\mu|\pi}. \quad (38)$$

In (36) and (38), the functions $\psi_k(\vec{\rho})$ and $f(\varphi)$ are given by formulas (23), (24) and present solutions of the problem for an infinitely thin solenoid; the coefficients b_m are given by

$$b_m = \frac{B_m}{A_m}. \quad (39)$$

Using the notation

$$F_{m,k}(\rho) = \rho^{|m|} \tilde{F}_{m,k}(\rho), \quad \tilde{F}_{m,k}(0) = 1, \quad (40)$$

one can present b_m as follows:

$$b_m = \frac{J_{|m+\mu|}(ka)}{(|m| + |m+\mu|) N_{|m+\mu|}(ka)} \cdot \frac{|m+\mu| - |m| - \frac{a\tilde{F}'_{m,k}(a)}{\tilde{F}_{m,k}(a)} - \frac{akJ_{|m+\mu|+1}(ka)}{J_{|m+\mu|}(ka)}}{1 + \frac{1}{|m|+|m+\mu|} \left[\frac{a\tilde{F}'_{m,k}(a)}{\tilde{F}_{m,k}(a)} - \frac{akN_{|m+\mu|-1}(ka)}{N_{|m+\mu|}(ka)} \right]}. \quad (41)$$

Let us take into account an estimation for the Bessel function [8], which is implied by its representation as a series:

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} (1 + \theta), \quad (42)$$

$$|\theta| < \exp \left\{ \frac{|z|^2}{4(1+\nu_0)} \right\} - 1, \quad \nu_0 = \min(|\nu+1|, |\nu+2|, \dots). \quad (43)$$

In a similar way, one can obtain an estimation for N_ν , $\nu \geq 0$:

$$N_\nu(z) = -\frac{2^\nu \Gamma(\nu)}{\pi z^\nu} (1 + \theta_1), \quad \nu \neq 0, \quad (44)$$

$$N_0(z) = \frac{2}{\pi} \ln \frac{z}{2} (1 + \theta_2), \quad (45)$$

where θ_1 and θ_2 are bounded (uniformly in m , $\nu = |m+\mu|$, with a fixed μ) functions in any bounded region z ; $\theta_i \rightarrow 0$ as $|z| \rightarrow 0$, while $|\theta_1| < C|z|^2$ at $\nu > 1$, the constant C being independent of m , $|\theta_1| \sim |z|^2 \ln |z|$ at $\nu = 1$, $|\theta_1| \sim |z|^{2\nu}$, at $0 < \nu < 1$, $|\theta_2| \sim 1/\ln |z|$.

From (39), (41), (34), (35), it is easy to see that all $b_m \rightarrow 0$ as $a \rightarrow 0$. Suppose, furthermore, that the quantities

$$\frac{a\tilde{F}'_{m,k}(a)}{\tilde{F}'_{m,k}(a)} \quad (46)$$

are uniformly bounded in m and a at $ka < \varepsilon$ for a sufficiently small ε . Then (41) and (42)–(45) imply an estimation⁵ for b_m :

$$b_m = \frac{\pi (ka)^{2|m+\mu|}}{2^{2|m+\mu|} (|m| + |m + \mu|) \Gamma(|m + \mu|) \Gamma(|m + \mu| + 1)} Q_{m,k}(a), \quad (47)$$

where the quantities $Q_{m,k}(a)$ are uniformly bounded in m and a for $ka < \varepsilon$.

With the help of (47), we easily find

$$|\phi_k(\varphi) - f(\varphi)| \leq 2 \sum_{|m+\mu| \leq 1} |b_m| + a^2 \theta(a), \quad \varphi - \delta \neq 0, \quad (48)$$

$\theta(a)$ being a bounded function as $a \rightarrow 0$.

Therefore, $\phi_k(\bar{\varphi}) \rightarrow \bar{f}(\varphi)$ as $a \rightarrow 0$ for all φ , $\varphi - \delta \neq 0$, at any value of μ .

It remains to prove that the values $a\tilde{F}'_{m,k}(a)/\tilde{F}_{m,k}(a)$ are uniformly bounded in m and a at $ka < \varepsilon$. This is done in Appendix B.

Let us examine the quantity $|\Psi_k - \psi_k|$. Using estimations (44), (45) and (47), we obtain, for any bounded region on the plane xy at $\rho \geq a$:

$$\begin{aligned} |\Psi_k - \psi_k| &\sim \left| \sum_{|m+\mu| < 1} b_m N_{|m+\mu|}(k\rho) \right| + \sum_{|m+\mu| \geq 1} \left(\frac{ka}{2} \right)^{|m+\mu|} \frac{1}{(|m| + |m + \mu|) \Gamma(|m + \mu| + 1)} \left(\frac{a}{\rho} \right)^{|m+\mu|} \\ &\sim \left| \sum_{|m+\mu| < 1} b_m N_{|m+\mu|}(k\rho) \right| + ka\phi\left(\frac{a}{\rho}\right) \sim \text{const} \cdot a^{\Delta\mu} + \text{const} \cdot a^{1-\Delta\mu} + \text{const} \cdot a, \end{aligned} \quad (49)$$

where $\phi\left(\frac{a}{\rho}\right)$ is a bounded (in a bounded part of the plane, at $\rho \geq a$) function is uniform in a as $a \rightarrow 0$.

Let us further use estimations (44), (45), (47), (53) and obtain for $\rho \leq a$:

$$|\Psi_k - \psi_k| \sim \left| \sum_{|m+\mu|} J_{|m+\mu|}(k\rho) \right| + \text{const} \cdot a \sim \text{const} \cdot a^{\Delta\mu} + \text{const} \cdot a^{1-\Delta\mu} + \text{const} \cdot a, \quad (50)$$

$$|\Psi_k(\vec{\rho})| \sim \sum_m \frac{1}{|m| + |m + \mu|} \cdot \frac{1}{N_{|m+\mu|}(ka)} \left(\frac{\rho}{a} \right)^{|m|} \rightarrow 0, \quad \text{as } a \rightarrow 0. \quad (51)$$

Estimations (49), (50) and (51) lead to the following result:⁶

a) for $\Delta\mu \neq 0$, $\Psi_k(\vec{\rho})$ converges as $a \rightarrow 0$ to the function $\psi_k(\vec{\rho})$ uniformly in any bounded part of the plane xy ;

b) at $\Delta\mu = 0$ this convergence is uniform only if $\rho > \delta$ for an arbitrarily small but fixed δ . This can be observed, in particular, from the fact that $\Psi_k(0) \sim a^n$ and $\psi_k(0) \sim 1$. However, the quantity $|\Psi_k(\vec{\rho})|^2 dxdy \sim |\Psi_k(\vec{\rho})|^2 \rho d\rho$, being the probability, “uniformly” converges to the probability $|\psi_k(\vec{\rho})|^2 dxdy$ in the sense that $\int_{\Delta} (|\Psi_k|^2 - |\psi_k|^2) dxdy$ tends to zero as $a \rightarrow 0$ uniformly with respect to an arbitrary choice of the integration region $\Delta \subset D$, D being an arbitrary bounded region of the plane xy .

Thus, it has been proved that the probability distribution in the problem of electron scattering by a solenoid of radius a , with any distribution of magnetic field inside of it, converges uniformly as $a \rightarrow 0$ and fixed flow of magnetic field μ , in any limited region of the plane (in the integral sense that has been explained above) to the probability distribution in the problem of electron scattering by an infinitely thin solenoid. The scattering amplitude corresponding to a thick solenoid converges as $a \rightarrow 0$ to the scattering amplitude corresponding to a thin solenoid uniformly in the region $\varepsilon < \varphi - \delta < 2\pi - \varepsilon$ for any fixed $\varepsilon > 0$.

⁵(47) gives a correct estimation for $m + \mu \neq 0$. For $m + \mu = 0$, we have $b_{[\mu]} \sim 1/\ln a$. It is important, however, that $b_{[\mu]} \rightarrow 0$ as $a \rightarrow 0$.

⁶The first term in (49) is of order $a^{\Delta\mu} \left(\frac{a}{\rho}\right)^{\Delta\mu} + a^{1-\Delta\mu} \left(\frac{a}{\rho}\right)^{1-\Delta\mu} \sim a^{\Delta\mu} + a^{1-\Delta\mu}$ at $\Delta\mu \neq 0$. However, at $\Delta\mu = 0$ ($\mu = -n$) the first term in (49) is of order $\ln \rho / \ln a$. At a fixed ρ , it tends to zero as $a \rightarrow 0$; however, the same term is ~ 1 at $\rho = a$.

4 Electron scattering by the field of a semi-infinite thin solenoid

As a matter of fact, we shall examine, in the first place, the case of two semi-infinite infinitely thin solenoids, with the corresponding potential

$$A_k^{(S)}(\vec{r}) = -\mu \frac{\varepsilon_{3ki} r_i}{r} \left(\frac{1}{r-z} - \frac{1}{r+z} \right) = -\mu_1 \frac{\varepsilon_{3ki} r_i}{r} \frac{z}{\rho^2}, \mu_1 = 2\mu. \quad (52)$$

Potential (52) corresponds to the potential of a monopole in Schwinger's formulation [1]. The Hamiltonian of the problem is given by

$$H = \left(-i\partial_k + A_k^{(S)} \right) + V(r) = -\frac{1}{r^2} \partial_r (r^2 \partial_r) + \frac{1}{r^2} J^2 - \frac{\mu_1^2}{r^2} + V(r), \quad (53)$$

$$J^2 = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} [-\partial_\theta^2 - 2i\mu_1 \cos \theta \partial_\theta + \mu_1^2]. \quad (54)$$

We have added to the Hamiltonian the potential V (which in the case of purely electromagnetic interaction between the electron and monopole is to be set equal to zero).

We shall now construct a self-adjoint operator corresponding to the differential expression (54). Let us define the operator J_0^2 as the differential operator (54) with the following domain D_0 : 1) the functions $\psi(\theta, \varphi)$ from D_0 must be twice-differentiable; 2) ψ and $\partial_\varphi \psi$ must be periodic in φ with the period 2π ; 3) $\psi = 0$ at $\theta < \varepsilon$ and $\theta > \pi - \varepsilon$ for a certain ε , where ε may be different for different functions. Property (2) is necessary due to physical reasons, as well as due to the fact that the Hamiltonian be defined on these functions in Cartesian coordinates in a natural way, since the singularities of the Hamiltonian are located only on the z axis. Note also that the functions with properties (1) and (2) form the domain of a self-adjoint operator corresponding to the differential (in φ) expression of second order with constant coefficients, on the segment $[0, 2\pi]$. The eigenfunctions of this operator are $e^{-im\varphi}$, $m = 0, \pm 1, \pm 2 \dots$, which are usually selected due to physical reasons.

D_0 is dense in the space L_2 of functions on a unit sphere, so that J_0^2 is a symmetric operator. Besides, J_0^2 is a positive operator. Let us find the eigenfunctions U of the operator $(J_0^2)^*$. Representing the function U in the form $U(\theta, \varphi) = e^{-im\varphi} U_m(\theta)$, we find that $U_m(\theta)$ satisfies the equation

$$\frac{1}{\sin^2 \theta} [-\sin \theta \partial_\theta (\sin \theta \partial_\theta) + m^2 + \mu_1^2 - 2m\mu_1 \cos \theta] U_m(\theta) = \lambda U_m(\theta). \quad (55)$$

Let us recall that the eigenfunctions of the operator $(J_0^2)^*$ are not subject to any boundary conditions [4, 5]. Eq. (55) has an exact solution, and the eigenfunctions $U_{m,l}(\theta, \varphi)$ are given by

$$\begin{aligned} U_{m,l}^{(1)} &= N_{m,l}^{-1} e^{-im\varphi} \left(\frac{1-t}{2} \right)^{\frac{\alpha}{2}} \left(\frac{1+t}{2} \right)^{\frac{\beta}{2}} F \left(-l, l + \alpha + \beta + 1; 1 + \alpha; \frac{1-t}{2} \right), \\ U_{m,l}^{(2)} &= M_{m,l}^{-1} e^{-im\varphi} \left(\frac{1-t}{2} \right)^{\frac{\alpha}{2}} \left(\frac{1+t}{2} \right)^{\frac{\beta}{2}} F \left(-l, l + \alpha + \beta + 1; 1 + \beta; \frac{1+t}{2} \right), \end{aligned} \quad (56)$$

where

$$\begin{aligned} \lambda &= \left(l + \frac{\alpha + \beta}{2} \right) \left(l + \frac{\alpha + \beta}{2} + 1 \right), \\ \alpha &= |-m + \mu_1|, \quad \beta = |m + \mu_1|, \quad t = \cos \theta. \end{aligned} \quad (57)$$

$F(a, b; c; x)$ is a hypergeometric function [9]; N and M are normalization coefficients (provided that the corresponding functions are normalizable). An analysis of the hypergeometric function $F(a, b; c; x)$ as $x \rightarrow 1$ shows that for a complex λ the functions $U_{m,l}^{(1)} \in L_2$ only on condition that $|m + \mu_1| < 1$, whereas $U_{m,l}^{(2)} \in L_2$ only on condition that $|-m + \mu_1| < 1$. Thus, for every complex-valued λ the operator $(J_0^2)^*$ has 4 eigenfunctions from L_2 for $\Delta\mu_1 \neq 0$ ($\mu_1 = -[\mu_1] - \Delta\mu_1$) and 2 such functions for $\Delta\mu_1 = 0$.

All essentially self-adjoint extensions J_T^2 of the operator J_0^2 are described [4, 5] by the following domains D_T :

$$D_T = D_0 + A_i U_i + A_i T_{ij} \overline{U_j}, \quad (58)$$

where U_i and \overline{U}_i are functions of the form (56) for some fixed complex l :

$$\begin{aligned} U_{1,2} &= U_{mi,l}^{(2)}, \quad m_1 = -[\mu_1], \quad m_2 = -[\mu_1] - 1, \\ U_{3,4} &= U_{mi,l}^{(1)}, \quad m_3 = [\mu_1], \quad m_4 = [\mu_1] + 1, \quad \Delta\mu_1 > 0, \end{aligned} \quad (59)$$

$$U_1 = U_{[\mu_1],l}^{(1)}, \quad U_2 = U_{-[\mu_1],l}^{(2)}, \quad \Delta\mu_1 = 0. \quad (60)$$

The functions \overline{U}_i are obtained from the functions U_i by the change $l \rightarrow l^*$, T_{ij} being an arbitrary (however fixed for a given extension) unitary matrix of dimension 4×4 for $\Delta\mu_1 \neq 0$ and 2×2 for $\Delta\mu_1 = 0$.

The requirement of “minimal singularity” for the self-adjoint extension (which we further denote as J^2) of the operator J_0^2 , that is, the condition that the domain of J^2 should contain only non-singular functions uniquely determines the extension (the matrix T).

We do not consider the case $\Delta\mu_1 \neq 0$. Let us only observe that all the functions from the domain of J^2 turn to zero along the z axis (at least, as $\theta^{\Delta\mu_1}$ or $\theta^{1-\Delta\mu_1}$, when $\theta \rightarrow 0$, and $(\pi - \theta)^{\Delta\mu_1}$ or $(\pi - \theta)^{1-\Delta\mu_1}$, when $\theta \rightarrow \pi$). For any other extension of the operator J_0^2 they either turn to zero or are singular on the z axis. In any case, this means that for $\Delta\mu_1 \neq 0$ the electron experiences scattering by the string.

We shall now examine the case $\Delta\mu_1$ ($\mu_1 = -n$). The matrix T for a “non-singular” extension is determined, as has been observed, in a unique way:

$$T_{kj} = \frac{n+i}{n-i} \delta_{kj}. \quad (61)$$

It is easy to see that the eigenfunctions of the operator J^2 from L_2 for this extension are given by (56), where l is allowed to take only positive integer values (for any other real l functions (61) are singular at $\theta = 0$ or at $\theta = \pi$):

$$l = 0, 1, 2, \dots \quad (62)$$

(another possible set $l + \alpha + \beta + 1 = 0, -1, -2, \dots$ gives the same set of eigenfunctions and eigenvalues). Therefore, J^2 is a positive operator.

For integer l , the hypergeometric functions

$$F\left(-l, l + \alpha + \beta + 1; 1 + \alpha; \frac{1-t}{2}\right), \quad F\left(-l, l + \alpha + \beta + 1; 1 + \beta; \frac{1+t}{2}\right)$$

are proportional to each other, as well as to the Jacobi polynomials [9] $P_n^{\alpha,\beta}(t)$ (thus, for integer l the solutions $U_{m,l}^{(1)}$ and $U_{m,l}^{(2)}$ coincide; the second independent solution in this case is singular at one of the ends $\theta = 0$ or $\theta = 2\pi$). Since in the case of fixed α and β the Jacobi polynomials form a complete orthonormal system in the space L_2 of functions on the segment $[-1, 1]$, with a scalar product determined by the weight $(1-t)^{-\alpha}(1+t)^{\beta}$, we find as a result that the eigenfunctions $U_{m,l}$ of the operator J^2 ,

$$U_{m,l} = N_{m,l}^{-1} e^{-im\varphi} \left(\frac{1-t}{2}\right)^{\frac{\alpha}{2}} \left(\frac{1+x}{2}\right)^{\frac{\beta}{2}} F\left(-l, l + \alpha + \beta + 1; 1 + \alpha; \frac{1-t}{2}\right), \quad (63)$$

form a complete system of vectors in the space of functions on a unit sphere.

This means that functions (63) form a complete system of (generalized) eigenvectors of the operator J^2 , whose spectrum is discrete and consists of the points

$$\lambda_L = L(L+1), \quad L \geq |\mu_1|, \quad (64)$$

with each value λ_L being $(2L+1)$ -times degenerate in m ($-L \leq m \leq L$).

Functions (63) coincide (up to a phase multiplier) with the generalized spherical functions $T_L^{m,\mu_1}(\varphi, \theta, 0)$ (see [10]). The angular form of solution (63) and its relation with the group of rotations has been known for a long time (see [11, 12] as well as the discussion in [1, 13]), and we will not dwell on this subject. Let us only note, once again, that, as has been proved, the operator J^2 , determined by its eigenvectors and spectrum (63) (64), is the only self-adjoint extension (of the differential expression (54)) with a “non-singular” domain.

Let us now construct solutions of the Schrödinger equation with Hamiltonian (53) (where J^2 is to be understood as the chosen self-adjoint extension) and with the asymptotic condition

$$\psi_k(\vec{r}) \rightarrow e^{i\vec{k}\vec{r}} + \frac{f(\theta, \varphi)}{r} e^{ikr}, \quad (65)$$

where $E = k^2$ is the energy of the electron.

The general solution of the Schrödinger equation has the form

$$\psi_k(\vec{r}) = \sum_{m,L} C_{m,L} T_L^{m,\mu_1}(\varphi, \theta, 0) R_{L,k}(r) i^L, \quad (66)$$

where $C_{m,L}$ are arbitrary coefficients, while the functions $R_{L,k}$ obey the equation

$$\left[\frac{1}{r^2} \partial_r (r^2 \partial_r) - \frac{L(L+1)}{r^2} - V_1(r) + k^2 \right] R_{L,k}(r) = 0, \quad (67)$$

$$V_1(r) = V(r) - \frac{\mu_1^2}{r^2} \quad (68)$$

and have the following asymptotics as $r \rightarrow \infty$:

$$R_{L,k}(r) \rightarrow \frac{1}{kr} \sin\left(kr - \frac{\pi}{2}L + \delta_L\right). \quad (69)$$

Since the functions T_L^{m,μ_1} form a complete system on a sphere, the asymptotics of the function $e^{i\vec{k}\vec{r}}$ can be presented in the form⁷ (see Appendix C):

$$\begin{aligned} e^{i\vec{k}\vec{r}} \xrightarrow{r \rightarrow \infty} & \frac{1}{2ikr} \left[e^{ikr} \sum_{m,L} T_L^{*m,\mu_1}(\varphi_k, \theta_k, 0) T_L^{m,\mu_1}(\varphi, \theta, 0) (2L+1) \right. \\ & \left. - e^{-ikr} \sum_{m,L} T_L^{*m,\mu_1}(\pi + \varphi_k, \pi - \theta_k, 0) T_L^{m,\mu_1}(\varphi, \theta, 0) (2L+1) \right], \end{aligned} \quad (70)$$

where φ_k, θ_k are the angular coordinates of the vector \vec{k} .

As a result, we find that the coefficients $C_{m,L}$ have the form

$$C_{m,L} = (-)^L (2L+1) e^{i\delta_L} T_L^{*m,\mu_1}(\pi + \varphi_k, \pi - \theta_k, 0). \quad (71)$$

Let us take into account the following property of the functions T_L^{m,μ_1} :

$$T_L^{*m,\mu_1}(\varphi, \theta, \psi) = T_L^{m,\mu_1}(\pi - \psi, \theta, \pi - \varphi), \quad (72)$$

and the addition formula for spherical functions [10]. Then, the wave function can be presented in the form⁸

$$\psi_k(\vec{r}) = e^{i\mu_1\Omega_1 - i\mu_1\pi} \sum_L (2L+1) i^L P_L^{-\mu_1,\mu_1}(\theta_{kr}) e^{i\delta_L} R_{L,k}(r), \quad (73)$$

$$\tan \frac{\Omega_1}{2} = \frac{\sin \frac{\theta + \theta_k}{2}}{\sin \frac{\theta - \theta_k}{2}} \tan \frac{\varphi - \varphi_k}{2}, \quad \cos \theta_{kr} = \frac{\vec{k}\vec{r}}{kr}. \quad (74)$$

For convenience, let us present the expression (73) for the wave function in the coordinate system related to the old coordinate system by the Euler angles $(\frac{\pi}{2} - \varphi_k, \theta_k, \alpha + \frac{\pi}{2})$. In the new coordinate system, the z axis is directed along the vector \vec{k} , whereas the potential has the form

$$\begin{aligned} \vec{A}^{(S,n)}(\vec{r}) &= \mu_1 \frac{\vec{n} \times \vec{r}}{r} \cdot \frac{\vec{n}\vec{r}}{r^2 - (\vec{n}\vec{r})^2}, \\ \vec{n} &= (\sin \theta_k \cos \alpha, \sin \theta_k \sin \alpha, \cos \theta_k) \end{aligned} \quad (75)$$

⁷The functions T_L^{m,μ_1} are normalized on a sphere by $\sqrt{\frac{4\pi}{2L+1}}$ exactly as the usual Legendre polynomials.

⁸The functions $T_L^{m,\mu}$ have the form [10] $T_L^{m,\mu}(\varphi, \theta, \psi) = e^{-im\varphi - i\mu\psi} P_L^{m,\mu}(\theta)$.

(the string of the potential is directed along the vector \vec{n}) and the wave function is given by the following expression:

$$\psi_k(\vec{r}) = e^{i\mu_1\Omega_1 - i\mu_1\alpha} \psi_k^{(0)}(\vec{r}) , \quad (76)$$

$$\psi_k^{(0)}(\vec{r}) = e^{-i\mu_1\pi + i\mu_1\varphi} \sum_L (2L+1) i^L e^{i\delta_L} P_L^{-\mu_1, \mu_1}(\theta_{kr}) R_{k,L}(r) , \quad (77)$$

$$\tan \Omega = \frac{\sin \theta_k \sin(\varphi - \alpha)}{\sin \theta_k \cos \theta \sin(\varphi - \alpha) - \sin \theta \cos \theta_k} . \quad (78)$$

Thus, the wave function (76) is a solution of the Schrödinger equation for the scattering of an electron by potential (75) with the initial momenta of the electron directed along the z axis. In addition, the wave function (77) is a solution of the same problem in the case of a string directed along the z axis. The scattering amplitude has the form ($\theta \neq 0, \pi$)

$$f(\theta, \varphi) = e^{i\mu_1\Omega_1 - i\mu_1\alpha} f^{(0)}(\theta, \varphi) , \quad (79)$$

$$f^{(0)}(\theta, \varphi) = \frac{e^{-i\mu_1\pi + i\mu_1\varphi}}{2ik} \sum_L (2L+1) P_L^{-\mu_1, \mu_1}(\theta) e^{2i\delta_L} , \quad (80)$$

where $f^{(0)}$ is the amplitude of electron scattering by the potential with a string directed along the z axis.⁹

Therefore, as has been said in Introduction, the dependence of the wave function and scattering amplitude on the direction of a string (on \vec{n}) enters only a non-essential phase multiplier. The cross-section of scattering, however, does not depend on the direction of a string.

A solution of the (seeming) paradox indicated in Introduction and related with the appearance of the phase multiplier $e^{-i\mu_1(\Omega-\alpha)}$ consists in the equality

$$\vec{A}^{(S,n)} - \vec{A}^{(S)} = ie^{-i\mu_1(\Omega-\alpha)} \vec{\partial} e^{i\mu_1(\Omega-\alpha)} , \quad (81)$$

which is valid only for integer μ_1 . Given this, the relation $\vec{A}^{(S,n)} - \vec{A}^{(S)} = -\mu_1 \vec{\partial} \Omega$ neither holds true nor follows from (81), since Ω is a discontinuous function. Representation (81) is nevertheless sufficient for us to transform the difference of potentials $\vec{A}^{(S,n)} - \vec{A}^{(S)}$ out of the Schrödinger equation with the help of a gauge (however, not a gradient one) phase transformation of the form (29).

Let us briefly examine the problem of electron scattering by the Dirac potential

$$A_k^{(D)}(\vec{r}) = -\mu \frac{\varepsilon_{3ki} r_i}{r(r-z)} . \quad (82)$$

One can easily see that the expression J^2 takes the form

$$J^2 = -\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \left[(-i\partial_\varphi + \mu)^2 + 2\mu (-i\partial_\varphi + \mu) \cos \theta + \mu^2 \right] , \quad (83)$$

and that its eigenfunctions are given by expressions (56), with the following parameters α, β :

$$\alpha = |-m + 2\mu| , \quad \beta = |m| . \quad (84)$$

Repeating literally the considerations presented in the case of the potential $\vec{A}^{(S)}$, we find that there exists a unique self-adjoint operator J^2 with a “non-singular” domain corresponding to the differential expression (83). If 2μ is non-integer, then the wave function turns to zero on the string, which implies that the electron “senses” the string.

Let us further examine the case of integer 2μ . A complete system of eigenfunctions $U_{m,l}(\theta, \varphi)$ of the operator J^2 is given by expression (63) with parameters (84), $l = 0, 1, 2, \dots$. We will present them in the form

$$U_{m,l}(\theta, \varphi) = e^{-im\varphi} P_L^{m', \mu}(\theta) = e^{-i\mu\varphi} T_L^{m', \mu}(\varphi, \theta, 0) , \quad (85)$$

$$m' = m - \mu , \quad L = l + \frac{|m|}{2} + \frac{|-m + 2\mu|}{2} \geq |\mu| . \quad (86)$$

⁹The expression (79) for the scattering amplitude was obtained by Zwanziger [2] with the help of the group-theory method.

By expanding the asymptotics of a plain wave in terms of an arbitrary complete system of functions on a sphere, we find that the wave function of electron scattering by the Dirac potential (83) is given by

$$\psi_k(\vec{r}) = \sum_{l,m} (-i)^L (2L+1) e^{i\delta_L + im(\varphi_k + \pi - \varphi)} P_L^{m',\mu}(\pi - \theta_k) P_L^{m',\mu}(\theta) R_{L,k}(r). \quad (87)$$

Using the addition rule for spherical functions, we present the expression for the wave function in the coordinate system related to the old coordinate system by Euler's angles $(\frac{\pi}{2} - \varphi_k, \theta_k, \alpha + \frac{\pi}{2})$:

$$\psi_k(\vec{r}) = e^{i\mu\Omega'} \sum_L i^L (2L+1) P_L^{-\mu,\mu}(\theta) e^{i\delta_L} R_{L,k}(r), \quad (88)$$

$$L = l + |\mu|, \quad l = 0, 1, 2, \dots,$$

$$\tan \frac{\Omega'}{2} = \frac{2 \sin \frac{\theta_k}{2} \cos \frac{\theta}{2} \sin(\varphi - \alpha)}{\sin \frac{\theta_k}{2} \cos \frac{\theta}{2} \cos(\varphi - \alpha) - \cos \frac{\theta_k}{2} \sin \frac{\theta}{2}}. \quad (89)$$

The corresponding amplitude ϕ equals to $(\theta \neq 0, \pi)$

$$\phi(\theta, \varphi) = \frac{e^{i\mu\Omega'}}{2ik} \sum_L (2L+1) P_L^{m',\mu}(\theta) e^{2i\delta_L}. \quad (90)$$

Expressions (88) and (90) present the wave function and scattering amplitude in the case of the potential (the z axis is directed along \vec{k})

$$\vec{A}^{(D,n)}(\vec{r}) = \mu \frac{\vec{n} \times \vec{r}}{r(r - \vec{n}\vec{r})} \quad (91)$$

and differ from the corresponding expressions in the case of potential (82) only by the phase multiplier $\exp(i\Omega')$. Therefore, in this problem the cross-section of scattering does not depend on the direction of the string, either. The dependence on the string appears only as a non-essential phase multiplier, which is related with the following equality:

$$\vec{A}^{(D,n)}(\vec{r}) - \vec{A}^{(D)}(\vec{r}) = ie^{-i\mu\Omega'} \vec{\partial} e^{i\mu\Omega'} \quad (92)$$

(valid only for integer 2μ).

Let us now compare solutions (77) and (88) in the case $\theta_k = 0, \alpha = 0$. It is clear that in the case of integer $\mu = \mu_1$ these solutions differ only by the phase multiplier $\exp(i\mu_1\varphi)$. At the same time, it is precisely this multiplier that connects the potentials of Dirac and Schwinger:

$$\vec{A}^{(S)}(\vec{r}) - \vec{A}^{(D)}(\vec{r}) = -\mu \frac{\vec{n}_z \times \vec{r}}{r^2 - z^2} = ie^{-i\mu\varphi} \vec{\partial} e^{i\mu\varphi}. \quad (93)$$

We arrive at the following conclusion: if the field of a monopole is described by several strings, then the magnetic charge (flow) of each string must be half-integer (for instance, it is only under this condition that an equality of the kind (92) holds true, which ensures the non-observability of a string). In the case of Schwinger's potential, we deal with two strings of equal charges, which implies the quantization condition $\mu = n$. In the case of Dirac's potential, there is only one string, and thus the quantization condition is $\mu = \frac{n}{2}$. If, for some reasons, we wish to describe the field of a monopole using k strings with equal charges, the quantization condition will then be $\mu = k \cdot \frac{n}{2}$.

5 Conclusion

Let us briefly state the results once again. In the case of electron scattering by the potential of infinitely thin infinite and semi-infinite solenoids, it has been demonstrated that in both of these problems there exist unique self-adjoint operators with "non-singular" domains, which, in view of physical reasons, must be identified with the Hamiltonians of the corresponding problems. In addition, if the magnetic charges of the strings (i.e., magnetic flows along the strings) do not conform to the quantization rules then the electron experiences scattering by the strings.

If, however, the magnetic charges actually conform to the quantization rules then the electron does not feel the presence of such a string. This result is also explained by the fact that the potential of a thin solenoid and the difference of the potentials of two half-solenoids can be presented in the form $U^{-1}\partial U$ and thus transformed out of the Schrödinger equation with the help of a gauge (however, not a gradient one) transformation of the form (29).

A Appendix

In [7], the problem of electron scattering by a thin solenoid is solved in the framework of a perturbation theory in $\Delta\mu$. The wave function is presented in the form ($\mu = n - \Delta\mu$)

$$\psi(\vec{\rho}) = \psi_0(\vec{\rho}) + \Delta\mu\psi_1(\vec{\rho}), \quad \psi_0(\vec{\rho}) = (-)^n e^{-in\varphi} e^{-i\vec{k}\vec{\rho}}, \quad (94)$$

where ψ_1 is subject to the equation

$$(-\Delta - k^2)\tilde{\psi}_1 = -\frac{2i}{\rho^2}\partial_\varphi e^{-i\vec{k}\vec{\rho}}, \quad \psi_1 = (-)^n e^{-in\varphi}\tilde{\psi}_1 \quad (95)$$

(the momentum \vec{k} being directed along the x axis). The corresponding scattering amplitude is

$$|f|^2 = \frac{\pi^2 \Delta\mu^2}{\tan^2 \frac{\varphi}{2}}; \quad (96)$$

however, the correct result for a small $\Delta\mu$ is

$$|f|^2 = \frac{\pi^2 \Delta\mu^2}{\sin^2 \frac{\varphi}{2}}. \quad (97)$$

The difference between (96) and (97) can be traced to the fact that (96) does not contain the contribution due to the partial wave $m = -n$, whereas this contribution is present in (97). To explain the absence of the wave $m = -n$ in (96) (and in ψ_1), let us represent the expansion ψ_1 in partial waves:

$$\psi_1(\vec{\rho}) = \sum_m e^{im\varphi - in\frac{\pi}{2}} i^m F_m(\rho). \quad (98)$$

Then, F_m obeys the equation

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho - \frac{(m+n)^2}{\rho^2} + k^2 \right) F_m(\rho) = -\frac{2(m+n)}{\rho^2} J_{m+n}(k\rho). \quad (99)$$

The solution of equation (99) with the asymptotics of a dispersed wave (regular at $\rho = 0$) is given by¹⁰

$$F_m(\rho) = \begin{cases} i\frac{\pi}{2} J_{m+n}(k\rho) - J_{m+n;\mu}(k\rho), & m+n > 0, \\ -i\frac{\pi}{2} J_{m+n}(k\rho) + (-)^{m+n} J_{|m+n|;\mu}(k\rho), & m+n < 0, \\ 0, & m+n = 0. \end{cases} \quad (100)$$

It is easy to see that (98) with the coefficient functions (100) is identical with the expression

$$\psi_1(\vec{\rho}) = (-)^{n+1} e^{in\varphi} \cdot 2i\Delta\mu \int d\vec{\rho}' G(\vec{\rho} - \vec{\rho}') \frac{1}{\rho'^2} \partial_{\varphi'} e^{i\vec{k}\vec{\rho}'} . \quad (101)$$

used in [7], where $G(\vec{\rho}) = \frac{i}{4} H_0^{(1)}(k\rho)$.

Let us compare (100) with the expressions for partial coefficients that follow from the expansion in $\Delta\mu$ of the exact solution (23). It is clear that in case $m+n \neq 0$ the expansion in $\Delta\mu$ of the exact solution leads precisely to expression (100). However, in case $m+n = 0$ the exact solution leads to

$$F_{-n}(\rho) = -i\frac{\pi}{2} H_0^{(1)}(k\rho). \quad (102)$$

¹⁰We have introduced the notation $J_{a;\mu}(x) = \frac{\partial}{\partial\mu} J_{a+\mu}(x) \Big|_{\mu=0}$.

At the same time, (100) implies $F_{-n} = 0$.

The solution of this paradox is given by the fact that the exact equation for F_m (as well as for ψ_k) cannot be solved by a perturbation theory in $\Delta\mu$.

Indeed, let us examine the equation for Bessel's functions $J_{\nu+\Delta\mu}$:

$$\left(\partial_x^2 + \frac{1}{x} \partial_x - \frac{(\nu + \Delta\mu)^2}{x^2} + 1 \right) J_{\nu+\Delta\mu}(x) = 0. \quad (103)$$

If one solves this equation in the framework of a formal perturbation theory in $\Delta\mu$, then the lowest correction in $\Delta\mu$ to J_0 : $J_{\Delta\mu} \approx J_0 + F_0$ must obey the equation

$$\left(\partial_x^2 + \frac{1}{x} \partial_x + 1 \right) F_0(x) = \frac{\Delta\mu^2}{x^2} J_0(x) = 0, \quad (104)$$

being a second-order equation in $\Delta\mu$. At the same time, the first derivative $\left. \frac{\partial}{\partial \Delta\mu} J_{\Delta\mu}(x) \right|_{\Delta\mu=0}$ does not vanish (and equals to $\frac{\pi}{2} N_0(x)$). This means that the r.h.s. of the equation for F_0 must contain terms of first-order in $\Delta\mu$. To construct a correct equation for F_0 , one can proceed, for instance, in the following way. Let us first present the equation for F_ε ($J_{\varepsilon+\Delta\mu} \approx J_\varepsilon + F_\varepsilon$, $F_\varepsilon \equiv \Delta\mu \frac{\partial}{\partial \varepsilon} J_\varepsilon$) and then proceed to the limit $\varepsilon \rightarrow 0$.

We obtain

$$\left(\partial_x^2 + \frac{1}{x} \partial_x + 1 \right) F_0(x) = \Delta\mu Q(x) = 0, \quad (105)$$

$$Q(x) = \lim_{\varepsilon \rightarrow 0} \left[\frac{2\varepsilon}{x^2} J_\varepsilon(x) - \frac{\varepsilon^2}{x^2} \frac{\partial}{\partial \varepsilon} J_\varepsilon(x) \right]. \quad (106)$$

From a simple calculation, it follows that

$$Q(x) = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{x^{2-\varepsilon}} = \frac{1}{x} \delta(x), \quad (107)$$

where the r.h.s. of (107) is understood as

$$\int dx dy f(\vec{x}) Q(\rho) = 2\pi f(0) \int_0^\infty \rho d\rho Q(\rho) = 2\pi f(0). \quad (108)$$

Therefore, the r.h.s. of (105) does not vanish and the solution of the corresponding equation is precisely given by $F_0 = \Delta\mu \frac{\pi}{2} N_0$.

The presented analysis shows that the correct equation for the coefficient functions F_m is

$$\left(\partial_\rho^2 + \frac{1}{\rho} \partial_\rho - \frac{(m+n)^2}{\rho^2} + k^2 \right) F_m(\rho) = -\frac{2(m+n)}{\rho^2} J_{m+n}(k\rho) + \delta_{0,m+n} \frac{\delta(\rho)}{\rho}. \quad (109)$$

Accordingly, the correct equation for $\tilde{\psi}_1$ has the form

$$(-\Delta - k^2) \tilde{\psi}_1(\rho) = -\frac{2i}{\rho^2} \partial_\varphi e^{i\vec{k}\vec{r}} + \frac{1}{\rho} \delta(\rho). \quad (110)$$

The additional (as compared to (95)) term in the r.h.s. of (110) does not appear in the formal perturbation theory in $\Delta\mu$; however, it is precisely this term that leads to the appearance of the missing partial wave with $m = -n$. One can easily see that ψ_1 , obtained as a solution of (110), is in agreement with the result of expanding the exact solution (23) in $\Delta\mu$.

B Appendix

Let us now present a proof of the uniform boundedness of the quantity $a\tilde{F}'_{m,k}(a)/\tilde{F}_{m,k}(a)$ (formula (46)).

From (33) and (40), it follows that the function $\tilde{F}_{m,k}$ obeys the equation

$$\left(\partial_\rho^2 + \frac{2|m|+1}{\rho}\partial_\rho + k^2 - \frac{2m\mu}{\rho^2}f_a(\rho) - \frac{\mu^2}{\rho^2}f_a^2(\rho)\right)\tilde{F}_{m,k}(\rho) = 0, \quad (111)$$

$$\tilde{F}_{m,k}(0) = 1.$$

The regularizing function $f_a(\rho)$ will be subject to the conditions

$$\begin{aligned} f_a(a) &= 1, \quad f_a(0) = 0, \quad |\varphi_a(\rho)| \leq C, \quad 0 \leq \rho \leq a, \\ |af'_a(a)| &< C_1, \quad 0 \leq \rho \leq a, \quad \text{for all } a, \end{aligned} \quad (112)$$

where the constants C, C_1 do not depend on a ; besides,

$$\varphi_a(\rho) \equiv \frac{a}{\rho}f_a(\rho). \quad (113)$$

Conditions (112) can be met, for instance, by the regularizing function

$$f_a(\rho) \equiv f\left(\frac{\rho}{a}\right). \quad (114)$$

It can be verified immediately that the differential equation and initial condition (111) can also be satisfied by a solution (provided that it does exist) of the following integral equation:

$$\tilde{F}_{m,k}(x) = 1 + \frac{1}{a} \int_0^x dy \left(1 - \frac{y^{2|m|}}{x^{2|m|}}\right) \left[\frac{m}{|m|} \cdot \mu \varphi_a(y) + \frac{\mu^2}{2|m|} \frac{y}{a} \varphi_a^2(y) - \frac{ak^2y}{2|m|} \right] \tilde{F}_{m,k}(y). \quad (115)$$

Let us solve (115) by iterations:

$$\tilde{F}_{m,k}(x) = 1 + \sum_{n=1}^{\infty} Y_n(x), \quad (116)$$

$$\tilde{F}_{m,k}(x) = \frac{1}{a} \int_0^x dy \left(1 - \frac{y^{2|m|}}{x^{2|m|}}\right) \left[\frac{m}{|m|} \mu \varphi_a(y) + \frac{\mu^2}{2|m|} \frac{y}{a} \varphi_a^2(y) - \frac{ak^2y}{2|m|} \right] Y_{n-1}(y). \quad (117)$$

Since there exists the inequality

$$\left| \frac{m}{|m|} \mu \varphi_a(y) + \frac{\mu^2}{2|m|} \frac{y}{a} \varphi_a^2(y) - \frac{ak^2y}{2|m|} \right| < C_2, \quad 0 \leq x \leq a, \quad (118)$$

where C_2 is a certain constant, independent of m and a , we find that Y_n admits the following estimation:

$$|Y_n(x)| \leq \frac{1}{n!} \left(C_2 \frac{x}{a}\right)^n. \quad (119)$$

Therefore, the iteration series (116) converges absolutely, whereas a solution of equation (115) does exist and coincide with the required solution of the differential equation (116) (because the second solution of the differential equation (116) is singular at zero). In addition, $\tilde{F}_{m,k}$ satisfies the condition (uniform in m and a)

$$\left| \tilde{F}_{m,k}(x) \right| \leq \exp\left(C_2 \frac{x}{a}\right), \quad 0 \leq x \leq a. \quad (120)$$

Taking a first derivative of (116), we obtain

$$\tilde{F}'_{m,k}(x) = \frac{2|m|}{ax^{2|m|+1}} \int_0^x dy y^{2|m|} \left(\frac{m}{|m|} \mu \varphi_a(y) + \frac{\mu^2}{2|m|} \frac{y}{a} \varphi_a^2(y) - \frac{ak^2y}{2|m|} \right) \tilde{F}_{m,k}(y), \quad (121)$$

whence it follows that $a\tilde{F}'_{m,k}$ is also bounded uniformly in m and a :

$$\left| a\tilde{F}_{m,k}(x) \right| \leq C_2 \exp\left(C_2 \frac{x}{a}\right), \quad 0 \leq x \leq a. \quad (122)$$

Let us now introduce a function $\phi_{m,k}$,

$$\tilde{F}'_{m,k}(x) = \exp\left\{ \frac{m}{|m|} \cdot \frac{\mu}{a} \int_0^x dy \varphi_a(y) \right\} \phi_{m,k}(y). \quad (123)$$

The function $\phi_{m,k}$ obeys the differential equation

$$\left[\partial_x^2 + \left(\frac{2m}{|m|} \frac{\mu}{x} f_a(x) + \frac{2|m|+1}{x} \right) \partial_x + k^2 + \frac{m}{|m|} \frac{\mu}{x} f'_a(x) \right] \phi_{m,k}(x) = 0, \quad \phi_{m,k}(0) = 1, \quad (124)$$

and the integral equation

$$\phi_{m,k}(x) = 1 - \frac{1}{2ma} \int_0^x dy \left(1 - \frac{y^{2|m|}}{x^{2|m|}} \right) \left[\left(\frac{m}{|m|} ayk^2 + a\mu f'_a(y) \right) \phi_{m,k}(y) + \mu y \varphi_a(y) \phi'_{m,k}(y) \right]. \quad (125)$$

From (120) and (122), it follows that $|\phi_{m,k}(x)|$ and $|x\phi'_{m,k}(x)|$ are bounded by some constants independent of m and a . We can, therefore, check once again that solution (125) does exist and coincide with solution (124). For instance, the term with $\phi'_{m,k}$ in (125) can be associated with the inhomogeneous term of the equation (bounded by a constant independent of m and a), and we obtain, as a result, an equation of the kind (115), which allows one to carry out the same estimation.

Looking once again at (125), we find that $\phi_{m,k}$ can be presented in the form

$$\phi_{m,k} = 1 + \frac{1}{m} \theta(x), \quad (126)$$

where

$$|\theta(x)| = \frac{1}{2a} \left| \int_0^x dy \left(1 - \frac{y^{2|m|}}{x^{2|m|}} \right) \left[\left(ayk^2 \frac{m}{|m|} + a\mu f'_a(y) \right) \phi_{m,k}(y) + \mu y \varphi_a(y) \phi'_{m,k}(y) \right] \right| \leq C_3, \quad (127)$$

with C_3 being independent of m and a . Using (122), (123), (126) and (127), we finally conclude that the quantity $a\tilde{F}'_{m,k}(a)/\tilde{F}_{m,k}(a)$ is actually bounded uniformly in m and a .

With the help of (121), it is easy to make sure that $\tilde{F}'_{m,k}$ has the following asymptotics in m :

$$\tilde{F}_{m,k}(x) = \frac{\mu}{a} \cdot \frac{m}{|m|} \varphi_a(x) e^{\frac{\mu}{a} \cdot \frac{m}{|m|} \int_0^x dy \varphi_a(y)} + O\left(\frac{1}{m}\right), \quad m \rightarrow \infty. \quad (128)$$

Therefore, quantity (46) admits the estimation

$$U_{m,k}(a) \equiv a \frac{\tilde{F}'_{m,k}(a)}{\tilde{F}_{m,k}(a)} = \mu \cdot \frac{m}{|m|} + O\left(\frac{1}{m}\right). \quad (129)$$

This estimation can also be obtained as follows. The function $U_{m,k}$ obeys the differential equation

$$xU'_{m,k}(x) + 2|m|U_{m,k}(x) + U_{m,k}^2(x) = 2m\mu f_a(x) + \mu^2 f_a^2(x) - k^2 x^2. \quad (130)$$

If we suppose that $U_{m,k}(x)$ and $U'_{m,k}(x)$ are bounded as $m \rightarrow \infty$, then (130) implies (129).

All of the obtained results are confirmed by an explicit calculation for two functions f_a , in case equation (111) has a manifest solution:

$$\begin{aligned}
1) \quad f_a &= \frac{\rho}{a}, \\
\tilde{F}_{m,k}(\rho) &= e^{-\xi \sqrt{\mu^2 - a^2 k^2} \frac{\rho}{a}} \Phi \left(\frac{1}{2} + |m| + \lambda, 2|m| + 1; 2\xi \sqrt{\mu^2 - a^2 k^2} \frac{\rho}{a} \right), \\
\lambda &= \frac{|m\mu|}{\sqrt{\mu^2 - a^2 k^2}}, \quad \xi = \frac{|m\mu|}{m\mu}, \\
2) \quad f_a &= \frac{\rho^2}{a^2}, \\
\tilde{F}_{m,k}(\rho) &= e^{-\frac{|\mu|}{2} \frac{\rho^2}{a^2}} \Phi \left(\frac{1}{2} + \frac{|m|}{2} + \frac{m\mu}{2|\mu|} - \frac{a^2 k^2}{4|\mu|}, |m| + 1; \frac{|\mu|}{2} \frac{\rho^2}{a^2} \right),
\end{aligned}$$

where $\Phi(\alpha, \beta; x)$ is the degenerate hypergeometric function [9].

C Appendix

Let us now find the expansion for the asymptotics of a plain wave $e^{i\vec{k}\vec{r}}$. We examine the integral

$$\int d\Omega e^{i\vec{k}\vec{r}} f(\theta, \varphi), \quad r \rightarrow \infty, \quad (131)$$

and calculate it by the method of a stationary phase. There are two stationary points of the function $kr(\cos\theta\cos\theta_k + \sin\theta\sin\theta_k\cos(\varphi - \varphi_k))$, namely,

$$\varphi = \varphi_k, \quad \theta = \theta_k, \quad \text{and} \quad \varphi = \varphi_k + \pi, \quad \theta = \pi - \theta_k.$$

It is easy to see that

$$\int d\Omega e^{i\vec{k}\vec{r}} f(\theta, \varphi) = \frac{2\pi}{ikr} [e^{ikr} f(\theta_k, \varphi_k) - e^{-ikr} f(\pi - \theta_k, \varphi_k + \pi)], \quad r \rightarrow \infty. \quad (132)$$

(132) implies that the asymptotics of a plain wave can be presented in the form

$$e^{i\vec{k}\vec{r}} \xrightarrow{r \rightarrow \infty} \frac{2\pi}{ikr} [e^{ikr} \delta(\Omega - \Omega_{\vec{k}}) - e^{-ikr} \delta(\Omega - \Omega_{\vec{k}})], \quad (133)$$

where the δ -function in (132) is understood as the δ -function on a sphere.

Let us now suppose that we have a complete, orthonormalized system of functions $T_i(\theta, \varphi)$ on a sphere. Then, the asymptotics of a plain wave can be presented in the form

$$e^{i\vec{k}\vec{r}} \xrightarrow{r \rightarrow \infty} \frac{2\pi}{ikr} \left[e^{ikr} \sum_n T_n^*(\theta_k, \varphi_k) T_n(\theta, \varphi) - e^{-ikr} \sum_n T_n^*(\pi - \theta_k, \varphi_k + \pi) T_n(\theta, \varphi) \right]. \quad (134)$$

In the case $T_i(\theta, \varphi) = Y_{lm}(\theta, \varphi)$, we have the well-known expansion for the asymptotics of a plain wave in spherical functions.

Formula (134) allows one to present expressions for the wave function and scattering amplitude in the case of Dirac's and Schwinger's potentials for arbitrary μ as a series of the form (66), (71) and (87).

For the sake of completeness, let us also obtain the asymptotics of a plain wave in the two-dimensional case. To this end, we apply the method of a stationary phase to the integral

$$\int d\varphi e^{i\vec{k}\vec{r}} f(\varphi), \quad \rho \rightarrow \infty. \quad (135)$$

The function $k\rho\cos(\varphi - \varphi_k)$ has two stationary points: $\varphi = \varphi_k$ and $\varphi = \varphi_k + \pi$; a simple integration yields

$$\int d\varphi e^{i\vec{k}\vec{r}} f(\varphi) \xrightarrow{\rho \rightarrow \infty} \sqrt{\frac{2\pi}{k\rho}} [e^{ik\rho - i\frac{\pi}{4}} f(\varphi_k) + e^{-ik\rho + i\frac{\pi}{4}} f(\varphi_k + \pi)]. \quad (136)$$

Therefore, a two-dimensional plain wave has the following asymptotics:

$$e^{i\vec{k}\vec{r}} \xrightarrow{\rho \rightarrow \infty} \sqrt{\frac{2\pi}{k\rho}} \left[e^{ik\rho - i\frac{\pi}{4}} \delta(\varphi - \varphi_k) + e^{-ik\rho + i\frac{\pi}{4}} \delta(\varphi - \varphi_k - \pi) \right]. \quad (137)$$

If we have a complete orthonormalized system of functions $T_n(\varphi)$ on the segment $[0, 2\pi]$, then the expansion of the asymptotics of a two-dimensional plain wave can be presented in the form

$$e^{i\vec{k}\vec{r}} \xrightarrow{r \rightarrow \infty} \sqrt{\frac{2\pi}{k\rho}} \left[e^{ik\rho - i\frac{\pi}{4}} \sum_n T_n^*(\varphi_k) T_n(\varphi) - e^{-ik\rho + i\frac{\pi}{4}} \sum_n T_n^*(\varphi_k + \pi) T_n(\varphi) \right]. \quad (138)$$

In the case $T_n(\varphi) = \frac{1}{\sqrt{2\pi}} e^{in\varphi}$, we obtain Exp. (22).

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